Pergamon

Computers Math. Applic. Vol. 31, No. 10, pp. 31–44, 1996
Copyright©1996 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0898-1221/96 \$15.00 + 0.00

S0898-1221(96)00050-8

Simultaneous Approximation of the Feller Operator

G. A. ANASTASSIOU AND M. ZHOU
Department of Mathematical Sciences, The University of Memphis
Memphis, TN 38152, U.S.A.

(Received January 1996; accepted February 1996)

Abstract—In this paper, a quantitative estimate for the simultaneous approximation of a function and its derivatives by the Feller operator is established via a probabilistic approach. This covers the cases of some classical approximation operators such as the Bernstein, Szász, Baskakov, and Gamma operators.

Keywords—Feller operator, Simultaneous approximation, Probabilistic method.

1. INTRODUCTION

For a sequence of i.i.d. nonnegative r.v.'s $X_1, X_2, \dots,$ with $E[X_1] = x$, the Feller operator (cf. [1, p. 218]) is defined as

$$F_n(f,\ x) := E\left[f\left(\frac{S_n}{n}\right)\right] := \int_0^\infty f(t)\, dP\left(\frac{1}{n}S_n \le t\right), \tag{1}$$

where $S_n = \sum_{i=1}^n X_i$, $P((1/n)S_n \le t)$ is the distribution function of $(1/n)S_n$ and f is a continuous function.

The Feller operator F_n contains some well-known classical operators such as Bernstein, Szász, Baskakov, and Gamma operators as special cases, and has been studied by many authors about various approximation properties (see, e.g., [2–6] and their citations).

Our purpose here is to investigate the simultaneous approximation of a function and its derivatives by the Feller operator. A quantitative estimate is obtained by using probabilistic methods. Our general setting allows us to establish results similar to already known ones for the above mentioned specific classical operators, as well as to produce another new related result. This is demonstrated in Section 4.

2. MAIN RESULT

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X(x)$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$ with $E[X(x)] = x \ge 0$. The variance and the moment generating function of $X(x)$ will be denoted by $\sigma^2(x) := E[(X(x) - x)^2]$ and $\Psi^*_{X(x)}(t) := E[\exp(tX(x))]$, respectively. For each fixed x, let $X_n \equiv X_n(x)$, $n = 1, 2, \dots,$ be a sequence of independent r.v.'s identically distributed as $X(x)$. Define the corresponding Feller operator $F_n(f, x)$ as in (1). Denote by $\mathcal{Z}_+$ the set of all nonnegative integers, $D := \frac{\partial}{\partial x}$ and $\omega_A(f,\ \delta)$ the first modulus of continuity of function f in the interval $[0,\ A]$:

$$\omega_A(f,\ \delta) := \sup\{|f(u) - f(v)|;\ u,\ v \in [0,\ A],\ |u - v| \le \delta\}.$$

Our main result follows.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

THEOREM 1. *Let $r \in \mathcal{Z}_+$ and $A > 0$ be fixed. Suppose for each fixed t that*

$$P(X(x) > t) \in C^r((0, A)), \tag{2}$$

and there exist two positive constants M and α such that

$$\left|D^k P(X(x) > t)\right| \le M e^{-\alpha t} \tag{3}$$

uniformly for all $0 \le k \le r$ and $0 < x < A$.

Assume further that $f \in C^r([0, \infty))$ and

$$\left|f^{(k)}(t)\right| \le K e^{\beta t}, \qquad 0 \le k \le r, \tag{4}$$

for some constants β and $K > 0$.

Then, for each $x \in (0, A)$, there holds for large n that

$$\begin{aligned} \left|D^r F_n(f, x) - f^{(r)}(x)\right| \le{}& \frac{2M^r}{\alpha^r}(1 + \sigma(x))\omega_A\left(f^{(r)}, \frac{1}{\sqrt{n}}\right) \\ &+ \left\{r! K e^{\beta A} \sum_{k=1}^{r-1} k M^k \left(\frac{2}{\alpha}\right)^{k+1} + \frac{r(r-1)}{2} K e^{\beta A} + 1\right\} \frac{1}{n}. \end{aligned} \tag{5}$$

NOTE 1. By assumption (3), it is easy to see that $\Psi^*_{X(x)}(\alpha/2) < \infty$ and consequently $\sigma(x) < \infty$. So the l.h.s. of (5) is finite and tends to 0 as $n \to \infty$.

3. PROOF OF THEOREM 1

To prove Theorem 1, we need the following two lemmas.

LEMMA 1. *Under the hypotheses of Theorem 1, there holds for large n that*

$$\begin{aligned} D^r E\left[f\left(\frac{S_n(x)}{n}\right)\right] \\ = \sum_{k=0}^{r} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) a_{rk} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j), \end{aligned} \tag{6}$$

where $(n)_k := n \dots (n - k + 1)$ and $a_{rk} \equiv a_{rk}(x, t_1, \dots, t_k, X_1, \dots, X_k)$'s satisfy the recurrence relation

$$a_{rk} = D a_{r-1,k} + a_{r-1,k-1}\, DP(X_k(x) > t_k), \qquad (r > 0), \tag{7}$$

with initial condition $a_{00} = 1$ and the conventions $a_{k0} = 0$ for $k > 0$ and $a_{k,-1} = a_{k,k+1} = 0$ for $k \ge 0$.

Furthermore, we have

$$\underbrace{\int_0^\infty \cdots \int_0^\infty}_{k} a_{rk} \prod_{i=1}^{k} dt_i = \begin{cases} 1, & \text{if } k = r, \\ 0, & \text{if } 0 < k < r, \end{cases} \tag{8}$$

and

$$|a_{rk}| \le \begin{cases} M^r e^{-\alpha(t_1 + \cdots + t_r)}, & k = r, \\ r! M^k e^{-\alpha(t_1 + \cdots + t_k)}, & 0 < k < r. \end{cases} \tag{9}$$

PROOF. Consider the general function $a(y) \equiv a(y, t_1, \dots, t_k)$ satisfying $a(y) \in C^1((0, A))$ and

$$\left|D^i a(y)\right| \le M e^{-\alpha(t_1 + \cdots + t_k)}, \tag{10}$$

for $i = 0, 1$ and for all $y \in (0, A)$. If $k < r$ denote

$$I(y) := \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) a(y) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP\left(X_j(y) \le t_j\right),$$

then

$$\begin{aligned} DI(x) &= \lim_{y\to x} \frac{1}{y-x}(I(y) - I(x)) \\ &= \lim_{y\to x} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) \frac{a(y) - a(x)}{y - x} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP\left(X_j(y) \le t_j\right) \\ &+ \lim_{y\to x} \frac{1}{y-x} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) a(x) \prod_{i=1}^{k} dt_i \\ &\times \left[\prod_{j=k+1}^{n} dP\left(X_j(y) \le t_j\right) - \prod_{j=k+1}^{n} dP(X_j(x) \le t_j)\right] \\ &=: D_1 + D_2. \end{aligned} \tag{11}$$

Observe that when $n > 2\beta/\alpha$

$$\begin{aligned} \left| f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) Da(x) \right| &\le K e^{(\beta/n)(t_1+\cdots+t_n)} M e^{-\alpha(t_1+\cdots+t_k)} \\ &\le MKe^{-(\alpha/2)(t_1+\cdots+t_k)} e^{(\alpha/2)(t_{k+1}+\cdots+t_n)}, \end{aligned}$$

and

$$\begin{aligned} &\lim_{y\to x} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} e^{-(\alpha/2)(t_1+\cdots+t_k)} e^{(\alpha/2)(t_{k+1}+\cdots+t_n)} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP\left(X_j(y) \le t_j\right) \\ &\qquad = \left(\frac{2}{\alpha}\right)^k \left(\Psi^*_{X(x)}\left(\frac{\alpha}{2}\right)\right)^{n-k} < \infty, \end{aligned}$$

by the Lebesgue convergence theorem and using condition (3).

Now [7, p. 270, Proposition 11.18] implies that

$$D_1 = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) Da(x) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j). \tag{12}$$

NOTE 2. We will encounter several integral operations such as changing integral with limit or changing integration orders. Our conditions (2)–(4) and the exponential bounds of the integral functions will guarantee the validity of those operations, which can be taken care of similar to the above, and we will not go into further detail each time.

Back to (11), we have

$$\begin{aligned} D_2 = \sum_{j=k+1}^{n} &\lim_{y\to x} \frac{1}{y-x} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) a(x) \prod_{i=1}^{k} dt_i \\ &\times \left[\prod_{\mu=k+1}^{j} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu)\right. \end{aligned} \tag{13}$$

$$- \prod_{\mu=k+1}^{j-1} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j}^{n} dP(X_\nu(x) \le t_\nu) \Bigg] = \sum_{j=k+1}^{n} \lim_{y \to x} \frac{1}{y-x} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n-1} a(x)$$

$$\times \int_0^\infty f^{(k)} \left(\frac{t_1 + \cdots + t_n}{n} \right) d[P(X_j(y) \le t_j) - P(X_j(x) \le t_j)] \tag{13 (cont.)}$$

$$\times \prod_{i=1}^{k} dt_i \prod_{\mu=k+1}^{j-1} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu).$$

Notice that

$$\begin{aligned}
&\int_0^\infty f^{(k)} \left(\frac{t_1 + \cdots + t_n}{n} \right) d[P(X_j(y) \le t_j) - P(X_j(x) \le t_j)] \\
&= \int_0^\infty f^{(k)} \left(\frac{t_1 + \cdots + t_n}{n} \right) d[-P(X_j(y) > t_j) + P(X_j(x) > t_j)] \\
&= f^{(k)} \left(\frac{t_1 + \cdots + t_n}{n} \right) [-P(X_j(y) > t_j) + P(X_j(x) > t_j)] \Big|_{t_j=0^-}^{t_j=\infty} \\
&\quad + \int_0^\infty \frac{1}{n} f^{(k+1)} \left(\frac{t_1 + \cdots + t_n}{n} \right) [P(X_j(y) > t_j) - P(X_j(x) > t_j)] \, dt_j \\
&= \int_0^\infty \frac{1}{n} f^{(k+1)} \left(\frac{t_1 + \cdots + t_n}{n} \right) [P(X_j(y) > t_j) - P(X_j(x) > t_j)] \, dt_j.
\end{aligned} \tag{14}$$

Using (14) in (13), we get

$$\begin{aligned}
D_2 &= \sum_{j=k+1}^{n} \lim_{y \to x} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} \frac{1}{n} f^{(k+1)} \left(\frac{t_1 + \cdots + t_n}{n} \right) a(x) \\
&\quad \times \frac{[P(X_j(y) > t_j) - P(X_j(x) > t_j)]}{y-x} \, dt_j \\
&\quad \times \prod_{i=1}^{k} dt_i \prod_{\mu=k+1}^{j-1} dP(X_\mu(y) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu) \\
&= \sum_{j=k+1}^{n} \frac{1}{n} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k+1)} \left(\frac{t_1 + \cdots + t_n}{n} \right) a(x) \, DP(X_j(x) > t_j) \, dt_j \prod_{i=1}^{k} dt_i \\
&\quad \times \prod_{\mu=k+1}^{j-1} dP(X_\mu(x) \le t_\mu) \prod_{\nu=j+1}^{n} dP(X_\nu(x) \le t_\nu) \\
&= \frac{n-k}{n} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k+1)} \left(\frac{t_1 + \cdots + t_n}{n} \right) a(x) \, DP(X_{k+1}(x) > t_{k+1}) \\
&\quad \times \prod_{i=1}^{k+1} dt_i \prod_{j=k+2}^{n} dP(X_j(x) \le t_j),
\end{aligned} \tag{15}$$

the last step being true due to the fact that X_j's are identically distributed.

Combining (11),(12), and (15), we obtain

$$D \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)} \left(\frac{t_1 + \cdots + t_n}{n} \right) a(x) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j) \tag{16}$$

$$= \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1+\cdots+t_n}{n}\right) Da(x) \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j)$$
$$+ \frac{n-k}{n} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k+1)}\left(\frac{t_1+\cdots+t_n}{n}\right) a(x)\, DP(X_{k+1}(x) > t_{k+1}) \qquad (16\ (\text{cont.}))$$
$$\times \prod_{i=1}^{k+1} dt_i \prod_{j=k+2}^{n} dP(X_j(x) \le t_j).$$

Now we are ready to start the proof of (6). It is easy to show that

$$E\left[f\left(\frac{S_n(x)}{n}\right)\right] = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f\left(\frac{t_1+\cdots+t_n}{n}\right) \prod_{j=1}^{n} dP(X_j(x) \le t_j).$$

In (16), take $k=0,\ a(x) = a_{00} \equiv 1$, we get

$$DE\left[f\left(\frac{S_n(x)}{n}\right)\right] = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f'\left(\frac{t_1+\cdots+t_n}{n}\right) DP(X_1(x) > t_1)\, dt_1 \prod_{j=2}^{n} dP(X_j(x) \le t_j).$$

So (6) (with (7)) is true for $r = 0,\ 1$.

Suppose (6) is true for $r-1$; i.e.,

$$D^{r-1}E\left[f\left(\frac{S_n(x)}{n}\right)\right]$$
$$= \sum_{k=0}^{r-1} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1+\cdots+t_n}{n}\right) a_{r-1,k} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j).$$

Substituting $a(x)$ in (16) for each $a_{r-1,k}$, we derive

$$D^rE\left[f\left(\frac{S_n(x)}{n}\right)\right]$$
$$= \sum_{k=0}^{r-1} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1+\cdots+t_n}{n}\right) Da_{r-1,k} \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j)$$
$$+ \sum_{k=0}^{r-1} \frac{(n)_k}{n^k} \frac{n-k}{n} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k+1)}\left(\frac{t_1+\cdots+t_n}{n}\right) a_{r-1,k}\, DP(X_{k+1}(x) > t_{k+1}) \qquad (17)$$
$$\times \prod_{i=1}^{k+1} dt_i \prod_{j=k+2}^{n} dP(X_j(x) \le t_j)$$
$$= \sum_{k=0}^{r} \frac{(n)_k}{n^k} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1+\cdots+t_n}{n}\right) [Da_{r-1,k} + a_{r-1,k-1}\, DP(X_k(x) > t_k)]$$
$$\times \prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j),$$

and (6) follows for r.

To prove (8), we need the following facts. Since

$$x = \int_0^\infty t\,dP(X(x) \le t) = \int_0^\infty t\,d(-P(X(x) > t),$$

integration by parts yields

$$x = \int_0^\infty P(X(x) > t)\,dt.$$

Hence,

$$1 = \int_0^\infty DP(X(x) > t)\,dt, \tag{18}$$

and

$$0 = \int_0^\infty D^i P(X(x) > t)\,dt, \qquad \text{for } 2 \le i \le r. \tag{19}$$

By induction on r, it is not difficult to show that a_{rk} is the sum of the terms of the form $D^{i_1}P(X_1(x) > t_1)\dots D^{i_k}P(X_k(x) > t_k)$ with $i_1 + \dots + i_k = r > 0$. If $k < r$, then at least one of i_j's is greater than 1, so by (19)

$$\begin{aligned} &\underbrace{\int_0^\infty \cdots \int_0^\infty}_{k} D^{i_1}P(X_1(x) > t_1)\dots D^{i_k}P(X_k(x) > t_k)\prod_{i=1}^k dt_i \\ &\qquad = \prod_{j=1}^k \int_0^\infty D^{i_j}P(X_j(x) > t_j)\,dt_i = 0. \end{aligned} \tag{20}$$

If $k = r$, we notice that

$$\begin{aligned} a_{rr} &= Da_{r-1,r} + a_{r-1,r-1}\,DP(X_r(x) > t_r) = a_{r-1,r-1}\,DP(X_r(x) > t_r) = \cdots \\ &= DP(X_1(x) > t_1)\dots DP(X_r(x) > t_r), \end{aligned} \tag{21}$$

and so

$$\underbrace{\int_0^\infty \cdots \int_0^\infty}_{r} a_{rr}\prod_{i=1}^r dt_i = \prod_{i=1}^r \int_0^\infty DP(X_i(x) > t_i)\,dt_i = 1 \tag{22}$$

by (19). Now (20) and (22) establish (8).

Finally, we come to (9). Denote d_{rk} the number of terms of the form $D^{i_1}P(X_1(x) > t_1)\dots$ $D^{i_k}\;P(X_k(x) > t_k)$ in a_{rk} when a_{rk} is decomposed as the sum of such terms. Let

$$d_r = \max\{d_{rk};\ 0 \le k \le r\};$$

then $Da_{r-1,k}$ counts at most $kd_{r-1} \le (r-1)d_{r-1}$ such terms, and $a_{r-1,k-1}\,DP(X_k(x) > t_k)$ gives no more than d_{r-1} terms. By (7), we find that

$$d_r \le (r-1)d_{r-1} + d_{r-1} = rd_{r-1}.$$

Note that when $d_1 = 1$, we get

$$d_r \le r!. \tag{23}$$

Moreover, by condition (2) of Theorem 1

$$\left|D^{i_1}P(X_1(x) > t_1)\dots D^{i_k}P(X_k(x) > t_k)\right| \le M^k e^{-\alpha(t_1+\dots+t_k)}.$$

Together, we have for $k < r$ that

$$|a_{rk}| \le r!M^k e^{-\alpha(t_1+\dots+t_k)},$$

and by (21) $|a_{rr}| \le M^r e^{-\alpha(t_1+\dots+t_r)}$; therefore, (9) holds. ∎

LEMMA 2. *Under the hypotheses of Theorem 1, there holds for large n that*

$$\left|E\left[f\left(\frac{S_n(x)}{n}\right)\right]-f(x)\right| \le (1+\sigma(x))\omega_A\left(f,\frac{1}{\sqrt{n}}\right)+2Ke^{\beta A}(\rho_A(x))^n, \tag{24}$$

where $(\rho_A(x))^2 = \inf_{t>0} E[e^{t(X(x)-A)}] < 1$.

PROOF. We have

$$\begin{aligned}&\left|E\left[f\left(\frac{S_n(x)}{n}\right)\right]-f(x)\right|\\ &\le \int_0^A |f(t)-f(x)|\,dP\left(\frac{1}{n}S_n(x)\le t\right)+\int_A^\infty |f(t)-f(x)|\,dP\left(\frac{1}{n}S_n(x)\le t\right)\\ &:= R_1+R_2,\end{aligned} \tag{25}$$

$$\begin{aligned}R_1 &\le \int_0^A \omega_A(f,|t-x|)\,dP\left(\frac{1}{n}S_n(x)\le t\right)\\ &\le \omega_A\left(f,\frac{1}{\sqrt{n}}\right)\int_0^A (1+\sqrt{n}\,|t-x|)\,dP\left(\frac{1}{n}S_n(x)\le t\right)\\ &\le \omega_A\left(f,\frac{1}{\sqrt{n}}\right)\left(1+\sqrt{n}\left(E\left[\left(\frac{1}{n}S_n(x)-x\right)^2\right]\right)^{1/2}\right)\\ &= (1+\sigma(x))\omega_A\left(f,\frac{1}{\sqrt{n}}\right),\end{aligned} \tag{26}$$

and

$$R_2 \le 2K\int_A^\infty e^{\beta t}\,dP\left(\frac{1}{n}S_n(x)\le t\right) \le 2K\left(E\left[e^{(2\beta/n)S_n(x)}\right]\right)^{1/2}\left(P\left(\frac{1}{n}S_n(x)\ge A\right)\right)^{1/2}.$$

Furthermore, [8, Theorem 1] (see also [4, Lemma 3) leads to

$$P\left(\frac{1}{n}S_n(x)\ge A\right) \le (\rho_A(x))^{2n},$$

where $\rho_A(x)$ is as in (24). At the same time, [9, Theorem 3.1] implies that

$$E\left[e^{(2\beta/n)S_n(x)}\right] \le e^{2\beta x}\exp\left\{\frac{2(2\beta)^2\Psi^*_{X(x)}(\alpha/2)}{e^2 n((\alpha/2)-2\beta/n)^2}\right\}, \qquad \left(n>\frac{4\beta}{\alpha}\right). \tag{27}$$

Note here $\Psi^*_{X(x)}(\alpha/2)<\infty$ due to (2), and thus, when $n \ge \max\{64\beta\Psi^*_{X(x)}(\alpha/2)/e^2\alpha^2(A-x), 8\beta/\alpha\}$, we have

$$E\left[e^{(2\beta/n)S_n(x)}\right] \le e^{2\beta A}. \tag{28}$$

Hence, there holds for large n that

$$R_2 \le 2Ke^{\beta A}(\rho_A(x))^n. \tag{29}$$

Now (24) follows from (25),(26), and (29). ∎

PROOF OF THEOREM 1. In the case of $r=0$, inequality (5) can be easily derived from Lemma 2. We thus assume $r\ge 1$ in the following.

By (6) of Lemma 1, there hold

$$
\begin{aligned}
\left|D^r F_n(f,\ x) - f^{(r)}(x)\right| &= \left|D^r E\left[f\left(\frac{S_n(x)}{n}\right)\right] - f^{(r)}(x)\right| \\
&\le \frac{(n)_r}{n^r}\Bigg|\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(r)}\left(\frac{t_1+\cdots+t_n}{n}\right) a_{rr}\prod_{i=1}^{r} dt_i \\
&\quad \times \prod_{j=r+1}^{n} dP\left(X_j(x)\le t_j\right) - f^{(r)}(x)\Bigg| \\
&\quad + \left|\sum_{k=0}^{r-1}\frac{(n)_k}{n^k}\underbrace{\int_0^\infty \cdots \int_0^\infty}_{n} f^{(k)}\left(\frac{t_1+\cdots+t_n}{n}\right) a_{rk}\prod_{i=1}^{k} dt_i \right. \\
&\quad \left. \times \prod_{j=k+1}^{n} dP(X_j(x)\le t_j)\right| \\
&\quad + \left|1-\frac{(n)_r}{n^r}\right| \left|f^{(r)}(x)\right| \\
&:= I_1 + I_2 + I_3.
\end{aligned}
\tag{30}
$$

First we treat

$$
\begin{aligned}
I_1 &= \frac{(n)_r}{n^r}\Bigg|\underbrace{\int_0^\infty \cdots \int_0^\infty}_{r+1} f^{(r)}\left(\frac{t_1+\cdots t_r}{n} + \frac{n-r}{n}t\right) a_{rr}\prod_{i=1}^{r} dt_i \\
&\quad \times dP\left(\frac{1}{n-r}S_{n-r}(x)\le t\right) - f^{(r)}(x)\Bigg| \\
&\overset{(8)}{=} \frac{(n)_r}{n^r}\Bigg|\underbrace{\int_0^\infty \cdots \int_0^\infty}_{r+1} \left(f^{(r)}\left(\frac{t_1+\cdots t_r}{n} + \frac{n-r}{n}t\right) - f^{(r)}(x)\right) a_{rr}\prod_{i=1}^{r} dt_i \\
&\quad \times dP\left(\frac{1}{n-r}S_{n-r}(x)\le t\right)\Bigg| \\
&\overset{(9)}{\le} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{r+1} \left|f^{(r)}\left(\frac{t_1+\cdots t_r}{n} + \frac{n-r}{n}t\right) - f^{(r)}(x)\right| M^r e^{-\alpha(t_1+\cdots+t_r)}\prod_{i=1}^{r} dt_i \\
&\quad \times dP\left(\frac{1}{n-r}S_{n-r}(x)\le t\right) \\
&= \int_0^\infty\int_0^\infty \left|f^{(r)}\left(\frac{s}{n}+\frac{n-r}{n}t\right) - f^{(r)}(x)\right| \frac{1}{(r-1)!}M^r s^{r-1}e^{-\alpha s}\,ds \\
&\quad \times dP\left(\frac{1}{n-r}S_{n-r}(x)\le t\right), \qquad (\text{let } t_1+\cdots+t_r = s \text{ and } t_i = t_i \text{ for } i<r) \\
&= \int_0^{(A+x)/2}\int_0^{(A-x)n/2} \cdot + \int_0^{(A+x)/2}\int_{(A-x)n/2}^{\infty} \cdot + \int_{(A+x)/2}^{\infty}\int_0^\infty \cdot \ =: \ I_{11}+I_{12}+I_{13}.
\end{aligned}
\tag{31}
$$

There hold

$$
I_{11} \le \omega_A\left(f^{(r)},\lambda\right)\int_0^\infty\int_0^\infty \left\{1+\frac{1}{\lambda}\left(\frac{s}{n}+\frac{rt}{n}+|t-x|\right)\right\}\frac{1}{(r-1)!}M^r s^{r-1}e^{-\alpha s} \tag{32}
$$

$$\begin{aligned}
&\times ds\, dP\left(\frac{1}{n-r}S_{n-r}(x) \le t\right), \qquad (\lambda > 0) \\
&= \frac{M^r}{\alpha^r}\omega_A\left(f^{(r)}, \lambda\right)\left\{1 + \frac{1}{\lambda}E\left[\left|\frac{1}{n-r}S_{n-r}(x) - x\right|\right] + \frac{rx}{\lambda n} + \frac{r}{\lambda n\alpha}\right\} \\
&\le \frac{M^r}{\alpha^r}\omega_A\left(f^{(r)}, \frac{1}{\sqrt{n}}\right)\left\{1 + \frac{\sqrt{n}}{\sqrt{n-r}}\sigma(x) + \frac{rx}{\sqrt{n}} + \frac{r}{\alpha\sqrt{n}}\right\}, \qquad \left(\lambda = \frac{1}{\sqrt{n}}\right) \\
&\le \frac{2M^r}{\alpha^r}(1 + \sigma(x))\omega_A\left(f^{(r)}, \frac{1}{\sqrt{n}}\right),
\end{aligned} \tag{32 (cont.)}$$

for $n \ge (2rx)^2 + (2r/\alpha)^2 + 4r/3$. Also,

$$\begin{aligned}
I_{12} &\le \int_0^{(A+x)/2}\int_{(A-x)n/2}^{\infty} K\left(e^{\beta s/n+\beta t} + e^{\beta x}\right)\frac{1}{(r-1)!}M^r s^{r-1}e^{-\alpha s}\, ds\, dP\left(\frac{1}{n-r}S_{n-r}(x) \le t\right) \\
&\le \frac{2}{(r-1)!}M^r K e^{\beta A}\int_{(A-x)n/2}^{\infty} s^{r-1}e^{-(\alpha/2)s}\, ds, \qquad \left(n > \frac{2\beta}{\alpha}\right).
\end{aligned} \tag{33}$$

Using inequality (see [9, (3.6)]

$$s^{r-1} \le \left(\frac{4(r-1)}{e\alpha}\right)^{r-1} e^{(\alpha/4)s}, \qquad (s > 0),$$

it is straightforward to show that

$$I_{12} \le C\rho_1^n,$$

where $C = 8M^r K e^{\beta A}(4(r-1)/e\alpha)^{r-1}/\alpha(r-1)!$ and $\rho_1 = e^{-\alpha(A-x)/8} < 1$. Thus, when n is large enough, there holds

$$I_{12} \le \frac{1}{2n}. \tag{34}$$

Furthermore,

$$\begin{aligned}
I_{13} &\le \int_{(A+x)/2}^{\infty}\int_0^{\infty} 2Ke^{\beta s/n+\beta t}\frac{1}{(r-1)!}M^r s^{r-1}e^{-\alpha s}\, ds\, dP\left(\frac{1}{n-r}S_{n-r}(x) \le t\right) \\
&\le \frac{2}{(r-1)!}KM^r\int_0^{\infty} s^{r-1}e^{-(\alpha/2)s}\, ds\int_{(A+x)/2}^{\infty} e^{\beta t}\, dP\left(\frac{1}{n-r}S_{n-r}(x) \le t\right), \qquad \left(n > \frac{2\beta}{\alpha}\right) \\
&\le \frac{2^{r+1}KM^r}{\alpha^r}\left(E\left[e^{(2\beta/(n-r))S_{n-r}(x)}\right]\right)^{1/2}\left(P\left(\frac{1}{n-r}S_{n-r}(x) \ge \frac{A+x}{2}\right)\right)^{1/2} \\
&\le \frac{2^{r+1}KM^r}{\alpha^r}e^{\beta A}\left(\rho_{(A+x)/2}(x)\right)^{n-r},
\end{aligned}$$

when n is sufficiently large, which can be proved similarly as (29). Thus, when n is large enough, there holds

$$I_{13} \le \frac{1}{2n}. \tag{35}$$

In summary, we have for large n that

$$I_1 \le \frac{2M^r}{\alpha^r}(1 + \sigma(x))\omega_A\left(f^{(r)}, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}. \tag{36}$$

Next we are going to estimate I_2. By (9) of Lemma 1, we have for $k < r$ that

$$\left|\underbrace{\int_0^{\infty}\cdots\int_0^{\infty}}_{n} f^{(k)}\left(\frac{t_1 + \cdots + t_n}{n}\right) a_{rk}\prod_{i=1}^{k} dt_i \prod_{j=k+1}^{n} dP(X_j(x) \le t_j)\right|$$

$$= \left| \underbrace{\int_0^\infty \cdots \int_0^\infty}_{k+1} \left(f^{(k)} \left(\frac{t_1 + \cdots + t_k}{n} + \frac{n-k}{n} t \right) - f^{(k)} \left(\frac{n-k}{n} t \right) \right) a_{rk} \right.$$

$$\left. \times \prod_{i=1}^{k} dt_i dP \left(\frac{1}{n-k} S_{n-k}(x) \le t \right) \right|$$

$$\le \underbrace{\int_0^\infty \cdots \int_0^\infty}_{k+1} \left| f^{(k)} \left(\frac{t_1 + \cdots + t_k}{n} + \frac{n-k}{n} t \right) - f^{(k)} \left(\frac{n-k}{n} t \right) \right| r! M^k e^{-\alpha(t_1 + \cdots + t_k)}$$

$$\times \prod_{i=1}^{k} dt_i \, dP \left(\frac{1}{n-k} S_{n-k}(x) \le t \right)$$

$$= \int_0^\infty \int_0^\infty \left| f^{(k)} \left(\frac{s}{n} + \frac{n-k}{n} t \right) - f^{(k)} \left(\frac{n-k}{n} t \right) \right| \frac{r! M^k}{(k-1)!} s^{k-1} e^{-\alpha s}$$

$$\times ds \, dP \left(\frac{1}{n-k} S_{n-k}(x) \le t \right)$$

$$\le \int_0^\infty \int_0^\infty K e^{\beta(s/n) + \beta t} \frac{s}{n} \frac{r!}{(k-1)!} M^k s^{k-1} e^{-\alpha s} \, ds \, dP \left(\frac{1}{n-k} S_{n-k}(x) \le t \right)$$

(by mean value theorem and (4))

$$\le \frac{1}{n} K M^k k r! \left(\frac{2}{\alpha} \right)^{k+1} E \left[e^{(\beta/(n-k)) S_{n-k}(x)} \right] \le r! K e^{\beta A} k M^k \left(\frac{2}{\alpha} \right)^{k+1} \frac{1}{n},$$

when $n \ge (32\beta \Psi^*_{X(x)}(\alpha/2)/\alpha^2 e^2 (A - x)) + (4\beta/\alpha) + r$, similarly shown as (28). Therefore, when n is large enough, there holds

$$I_2 \le r! K e^{\beta A} \sum_{k=1}^{r-1} k M^k \left(\frac{2}{\alpha} \right)^{k+1} \frac{1}{n}. \tag{37}$$

Finally (cf. [10, p. 27])

$$I_3 \le \frac{r(r-1)}{2n} \left| f^{(r)}(x) \right| \le \frac{r(r-1)}{2n} K e^{\beta A}. \tag{38}$$

Now (5) follows from (30) and (36)–(38). ∎

4. APPLICATIONS

When specifying the underlying r.v.'s, the Feller operator (1) reduces to various concrete operators. We discuss four such operators in this section to demonstrate the applications of our general results.

EXAMPLE 1. (Bernstein operator) Let $X(x)$ have the Bernoulli distribution

$$P(X(x) = 1) = x, \quad P(X(x) = 0) = 1 - x, \qquad (0 < x < 1);$$

then (1) becomes the Bernstein operator

$$B_n(f, x) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Furthermore, for $0 < x < 1$

$$P(X(x) > t) = \begin{cases} 1, & t < 0, \\ x, & 0 \le t < 1, \\ 0, & t \ge 1, \end{cases}$$

$$DP(X(x)>t)=\begin{cases}0, & t<0,\\ 1, & 0\le t<1,\\ 0, & t\ge 1,\end{cases}$$

and $D^jP(X(x)>t)=0\ \ (j\ge 2)$. By (6) of Lemma 1, we have

$$\begin{aligned}
&\left|D^rB_n(f,\ x)-f^{(r)}(x)\right|\\
&\quad\le \frac{(n)_r}{n^r}\left|\underbrace{\int_0^1\cdots\int_0^1}_{n} f^{(r)}\left(\frac{t_1+\cdots+t_n}{n}\right)\right.\\
&\quad\quad\times\left.\prod_{i=1}^{r}dt_i\prod_{j=r+1}^{n}dP\left(X_j(x)\le t_j\right)-f^{(r)}(x)\right|+\left|\frac{(n)_r}{n^r}-1\right|\ \left|f^{(r)}(x)\right|\\
&\quad\le \frac{(n)_r}{n^r}\underbrace{\int_0^1\cdots\int_0^1}_{n}\left|f^{(r)}\left(\frac{t_1+\cdots+t_n}{n}\right)-f^{(r)}(x)\right|\prod_{i=1}^{r}dt_i\prod_{j=r+1}^{n}dP\left(X_j(x)\le t_j\right)\\
&\quad\quad+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|\\
&\quad=\frac{(n)_r}{n^r}\underbrace{\int_0^1\cdots\int_0^1}_{r}\int_0^1\left|f^{(r)}\left(\frac{t_1+\cdots+t_r}{n}+\frac{n-r}{n}t\right)-f^{(r)}(x)\right|\\
&\quad\quad\times\prod_{i=1}^{r}dt_i\,dP\left(\frac{1}{n-r}S_{n-r}(x)\le t\right)+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|\\
&\quad\le \frac{(n)_r}{n^r}\omega_1\left(f^{(r)},\lambda\right)\underbrace{\int_0^1\cdots\int_0^1}_{r}\int_0^1\left(1+\frac{1}{\lambda}\left|\frac{t_1+\cdots+t_r}{n}+\frac{n-r}{n}t-x\right|\right)\\
&\quad\quad\times\prod_{i=1}^{r}dt_i\,dP\left(\frac{1}{n-r}S_{n-r}(x)\le t\right)+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|\\
&\quad\le \frac{(n)_r}{n^r}\omega_1\left(f^{(r)},\lambda\right)\left\{1+\frac{r}{n\lambda}+\frac{rx}{n\lambda}+\frac{1}{\lambda}\left(E\left[\left(\frac{1}{n-r}S_{n-r}(x)-x\right)^2\right]\right)^{1/2}\right\}\\
&\quad\quad+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|\\
&\quad\le \frac{(n)_r}{n^r}\omega_1\left(f^{(r)},\frac{1}{\sqrt{n}}\right)\left\{1+\frac{r(1+x)}{\sqrt{n}}+\frac{\sqrt{x(1-x)}\sqrt{n}}{\sqrt{n-r}}\right\}\\
&\quad\quad+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|,\qquad\left(\lambda=\frac{1}{\sqrt{n}}\right)\\
&\quad=\left(1+\sqrt{x(1-x)}+a_n\right)\omega_1\left(f^{(r)},\frac{1}{\sqrt{n}}\right)+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|,
\end{aligned}$$

where

$$a_n:=\frac{(n)_r}{n^r}-1+\left(\frac{(n)_r}{n^r}\frac{\sqrt{n}}{\sqrt{n-r}}-1\right)\sqrt{x(1-x)}+\frac{(n)_r}{n^r}\frac{r(1+x)}{\sqrt{n}}\to 0,\qquad(n\to\infty);$$

i.e.,

$$\left|D^rB_n(f,\ x)-f^{(r)}(x)\right|\le\left(1+\sqrt{x(1-x)}+a_n\right)\omega_1\left(f^{(r)},\frac{1}{\sqrt{n}}\right)+\frac{r(r-1)}{2n}\left|f^{(r)}(x)\right|.$$

For the simultaneous approximation by the Bernstein operator B_n, see [10–12].

EXAMPLE 2. (Szász operator) Let $X(x)$ follow the Poisson distribution

$$P(X(x) = k) = e^{-x}\frac{x^k}{k!}, \qquad (k = 0, 1, 2, \dots);$$

then (1) becomes the Szász operator

$$S_n(f,\, x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{n^k}{k!}x^k.$$

Furthermore, we have

$$P(X(x) > t) = 1 - P(X(x) \le t) = e^{-x}\sum_{k=[t]+1}^{\infty}\frac{x^k}{k!},$$

and $DP(X(x) > t) = (1/[t]!)x^{[t]}e^{-x}$. From now on '[ ]' will denote the integer part function. So by Leibniz's formula, we derive

$$D^k P(X(x) > t) = x^{[t]}e^{-x}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{k-1-j}}{([t]-j)!}x^{-j}.$$

Now by Stirling's formula, we can find constants M and $\alpha > 0$ such that (3) holds. Therefore, by Theorem 1, we have

$$\left|D^r S_n(f,\, x) - f^{(r)}(x)\right| = O\left(\omega_A\left(f^{(r)},\, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\right), \qquad (n \to \infty),$$

for x and f as in Theorem 1.

The simultaneous approximation of the Szász operator has been studied by many authors. One can find related expositions in [13–15] and the papers cited therein.

EXAMPLE 3. (Baskakov operator) Let $X(x)$ have the geometric distribution

$$P(X(x) = k) = \frac{1}{1+x}\left(\frac{x}{1+x}\right)^k, \qquad (k = 0, 1, 2, \dots);$$

then (1) becomes the (special) Baskakov operator (cf. [4,6])

$$B_n^*(f,\, x) = (1+x)^{-n}\sum_{n=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k.$$

We have, for $0 < x < A$, that

$$P(X(x) > t) = \sum_{k=[t]+1}^{\infty}\frac{1}{1+x}\left(\frac{x}{1+x}\right)^k = \left(\frac{x}{1+x}\right)^{[t]+1}.$$

Also,

$$\begin{aligned}
\left|D^k P(X(x) > t)\right| &= \left|\sum_{j=0}^{k}\binom{k}{j}\left(D^j x^{[t]+1}\right)D^{k-j}(1+x)^{-([t]+1)}\right| \\
&\le \sum_{j=0}^{k}\binom{k}{j}([t]+1)\dots([t]-j+2)x^{[t]+1-j}([t]+1)\dots([t]+k-j)(1+x)^{-([t]+1-j)-k} \\
&\le ([t]+k)^k\sum_{j=0}^{k}\binom{k}{j}\left(\frac{x}{1+x}\right)^{[t]+1-j}\left(\frac{1}{1+k}\right)^k \\
&\le 2^r([t]+r)^r\left(\frac{A}{1+A}\right)^{[t]-r}, \qquad \text{for all } k \le r.
\end{aligned}$$

Now it is clear that there exist M and $\alpha > 0$ such that (3) holds. By Theorem 1, we have

$$\left|D^r B_n^*(f,\, x) - f^{(r)}(x)\right| = O\left(\omega_A\left(f^{(r)},\, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\right), \qquad (n \to \infty),$$

for x and f as in Theorem 1.

Note that the simultaneous approximation of general Baskakov operators has been studied in [15].

EXAMPLE 4. (Gamma operator) Let $X(x)$ follow the exponential distribution with density

$$g(v,\, x) = x^{-1} e^{-v/x}, \qquad v > 0, \quad 0 < a \le x \le b < \infty;$$

then (1) becomes the Gamma operator

$$G_n(f,\, x) = \frac{x^{-n}}{(n-1)!} \int_0^\infty f\left(\frac{v}{n}\right) v^{n-1} e^{-v/x}\, dv.$$

Now

$$P(X(x) > t) = \int_t^\infty \frac{1}{x} e^{-v/x}\, dv = e^{-t/x},$$

and it is easy to show that for $k \le r$, there exist b_{kj}'s such that

$$\begin{aligned} \left|D^k P(X(x) > t)\right| &= \left| \frac{1}{x^{2k}} \left(\sum_{j=0}^{k} b_{kj} x^j t^{k-j} \right) e^{-t/x} \right| \\ &\le \frac{(r+1) \max(|b_{kj}|; k \le r,\, j \le k)(\max(b,\, 1))^r}{(\min(a,\, 1))^{2r}} t^r e^{-(1/b)t}. \end{aligned}$$

Thus, we can find M and $\alpha > 0$ satisfying (3) and by Theorem 1, there holds

$$\left|D^r G_n(f,\, x) - f^{(r)}(x)\right| = O\left(\omega_A\left(f^{(r)},\, \frac{1}{\sqrt{n}}\right) + \frac{1}{n}\right), \qquad (n \to \infty),$$

for x and f as in Theorem 1.

Note that the Gamma operator G_n considered here is different from the Gamma operator of Müller (for its simultaneous approximation, see [15,16]), so to the best of our knowledge this result here is new.

REFERENCES

1. W. Feller, *An Introduction to Probability Theory and Its Applications II*, Wiley, New York, (1966).
2. L. Hahn, A note on stochastic methods in connection with approximation theorems for positive linear operators, *Pacific J. Math.* **101**, 307–319 (1981).
3. M.K. Khan, On the rate of convergence of Bernstein power series for functions of bounded variation, *J. Approx. Theory* **57**, 90–103 (1989).
4. R.A. Khan, Some probabilistic methods in the theory of approximation operators, *Acta Math. Acad. Sci. Hungar.* **35**, 193–203 (1980).
5. D. Stancu, Use of probabilistic methods in the theory of uniform approximation of continuous functions, *Rev. Roum. Math. Pures et Appl.* **14**, 673–691 (1969).
6. X. Xiong, The best asymptotic constant of a class of approximation operators, *J. Approx. Theory* **70**, 348–357 (1992).
7. R.H. Royden, *Real Analysis*, Macmillan, New York, (1988).
8. H. Chernoff, A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations, *Ann. Math. Stat.* **23**, 493–507 (1952).
9. D. Pfeifer, Approximation-theoretic aspects of probabilistic representations for operator semigroups, *J. Approx. Theory* **43**, 271–296 (1985).
10. H.-B. Knoop and P. Pottinger, Ein Satz vom Korovkin-Type für C^k-Raume, *Math. Z.* **148**, 23–32 (1976).

11. H.H. Gonska, Quantitative Korovkin type theorems on simultaneous approximation, *Math. Z.* **186**, 419–433 (1984).
12. R. Martini, On the approximation of functions together with their derivatives by certain linear positive operators, *Indag. Math.* **31**, 473–481 (1969).
13. P.N. Agrawal and H.S. Kasana, On simultaneous approximation by Szász-Mirakian operators, *Bull. Inst. Math. Academia Sinica* **22**, 181–188 (1994).
14. P.L. Butzer, On the extension of Bernstein polynomials to the infinite interval, *Proc. Amer. Math. Soc.* **5**, 547–553 (1954).
15. H.-B. Knoop and P. Pottinger, On simultaneous approximation by certain linear positive operators, *Arch. Math.* **48**, 511–520 (1987).
16. R.K.S. Rathore, Lipschitz-Nikolkii constants for the Gamma operators of Müller, *Math. Z.* **141**, 193–198 (1975).